A Design-Based Research Approach to Addressing Epistemological Obstacles in Introductory Proofs Courses

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As has been well-documented, the epistemological obstacles associated with teaching and learning mathematical proofs persist despite research-based instruction. We describe the ongoing design process of our NSF-funded project aimed at understanding and addressing those obstacles in introductory proofs courses, using proof by mathematical induction as an anchor. Our process is framed by two cycles of designed-based research. The first cycle corresponds to designing and implementing research-based instruction on mathematical induction, whereas the second cycle broadens the scope of our research to other introductory proofs topics. This paper reports on the outcomes of the first cycle, the transition between the first and second cycles, and the project's end products.

Keywords: designed-based research, epistemological obstacles, proofs, proof by mathematical induction, quantifiers.

There is a wide recognition among researchers and practitioners that proofs and proving are of vital importance in students' learning of mathematics at all levels of education (NCTM, 2000). At the same time, a growing body of research has identified numerous challenges associated with teaching and learning proofs (e.g., Brousseau, 1997; Brown, 2008; Dawkins & Weber, 2017; Hanna & de Villiers, 2012; Harel & Sowder, 2007; Shipman, 2016; Sierpińska, 1987; Stylianides, 2014, 2016). Sierpińska (1987) and Brousseau (1997) have conceptualized the necessary challenges in students' mathematical development in terms of epistemological obstacles (EOs). EOs in teaching and learning mathematical proofs are the primary focus of this project.

Here, we frame EOs as cognitive challenges that persist even in response to research-based instruction. Thus, EOs can be experienced both by students and teachers during instructional interactions. When instructors experience EOs, there is a tension between the desire to circumvent them and the need to provide students with opportunities to develop logical structures that are fundamental for proving. However, as it has been pointed out by Brousseau (2002), addressing an obstacle head on is essential for overcoming it because "it will resist being rejected and, as it must, it will try to adapt itself locally, to modify itself at the least cost, to optimize itself in a reduced field, following a well-known process of accommodation" (p. 85). As such, because EOs are persistent, students must internalize an intellectual need for the underlying concepts to motivate, persevere, and successfully overcome the obstacles.

Therefore, the overarching goals of our project are to 1) identify the EOs associated with teaching and learning mathematical proofs and 2) design instructional tools for evoking and addressing these obstacles. The research questions guiding our study are as follows:

1. What are the EOs associated with teaching and learning mathematical proofs?
2. What instructional tools are suitable for evoking and addressing these EOs?

In answering these questions, we employ a cyclic design-based research approach (Anderson & Shattuck, 2012; Bakker & Van Eerde, 2015; Cobb et al., 2003; Gravemeijer & Cobb, 2006). The first cycle was conducted in Spring 2018, when the third and fourth authors implemented research-based instruction on proof by mathematical induction (PMI) in their classrooms. The results are presented in Norton et al. (2022). We use PMI as an anchor for broadening the scope of our research. During the second cycle, the same teachers will use the designed instructional materials in their respective Introduction to Proofs classes – one in Fall 2022 and one in Spring 2023. This paper reports on the outcomes of the first cycle, the transition between the first and second cycles, and the project's end products (see Figure 1).

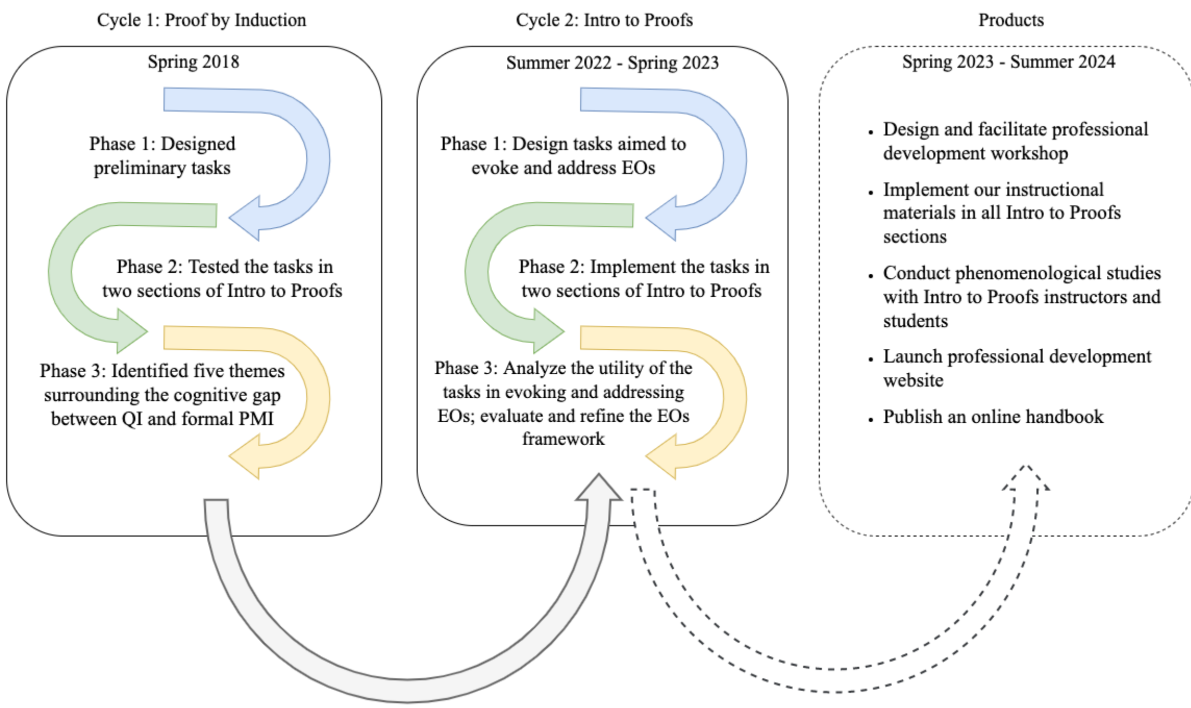

Figure 1: Two cycles and the end products of the project.

Literature Review and Theoretical Framework

Prior research has documented EOs on a variety of concepts studied in proofs-based courses. The following challenges have been associated with students' mastery of PMI: understanding the role of the base case (Baker, 1996; Ernst, 1984; Ron & Dreyfus, 2004; Stylianides et al., 2007), treating logical implication as an invariant relationship (Dubinsky, 1986, 1991; Norton & Arnold, 2017, 2019), discerning between the truth of the conjecture and the inductive hypothesis (Movshovirz-Hadar, 1993), attending to (hidden) quantifiers (Shipman, 2016) and the proper use of related language (Ernst, 1984; Movshovitz-Hadar, 1993; Stylianides et al., 2007), and having domain-specific knowledge particular to the conjecture (Dubinsky, 1991). Many of these challenges, especially quantification, extend well beyond PMI (Dawkins & Roh, 2020; Lew & Mejía-Ramos, 2019).

Instructional Approaches for Addressing EOs

Awareness of EOs should inform the design of instructional approaches to introductory proofs topics. Traditional instructional approaches may be insufficient in addressing these obstacles, necessitating the development of alternative instructional techniques that support student learning.

In the case of PMI, traditional instruction introduces the technique as a three-step procedure: (1) prove the base case, (2) assume the inductive hypothesis, and (3) prove the inductive step. However, this procedure can inadvertently cause learners to bypass the logic of PMI, ultimately circumventing experiencing necessary challenges (Harel, 2002).

An alternative approach that combines procedure with logical structure was first suggested by Avital and Libeskind (1978) and then further elaborated by Harel (2002) as "quasi-induction." A student who uses this method shows that $P(1)$ is true and that $P(1) \rightarrow P(2)$, $P(2) \rightarrow P(3)$, and so on. This leads to the plausible conclusion that eventually $P(n-1) \rightarrow P(n)$. Although formal PMI may be considered to be a natural generalization of quasi-induction, there is a significant cognitive gap between the two (Harel, 2002). Norton et al. (2022) implemented research-based instruction to address and better understand this gap.

Design-Based Research

Design-based research (DBR) is an interventionist approach aimed at weaving together educational practice and theory (Bakker & Van Eerde, 2015). DBR is typically used to develop educational materials and accompanying theoretical insights into how these materials can be used in practice.

Gravemeijer and Cobb (2006) framed DBR in terms of three interrelated phases: 1) preparing for the experiment, 2) experimenting in the classroom, and 3) conducting retrospective analysis. These phases occur in repeated cycles in which the last phase of the previous cycle informs the first phase of the following cycle, and so on. During the first phase, researchers scrutinize the problem of interest, synthesize the available research literature, curricula, and textbooks, and articulate the learning objectives and theoretical intents of the experiment. In the second phase, the experiment is conducted using the designed instructional materials as a guideline for teaching and observing. Data collection and preliminary data analysis also takes place during this phase. Once the experiment is complete and the data has been collected, the retrospective analysis begins in phase three. The comprehensive data sets must be analyzed systematically while simultaneously documenting the grounds for the subsequent cycle. As such, the retrospective analysis should examine the utility of the designed instructional materials and spark ideas for how they might be refined and complemented.

Cycle 1 (Spring 2018)

As aforementioned, prior research indicates a number of EOs experienced by introductory proofs students and suggests fruitful instructional approaches for supporting students in overcoming these obstacles. In particular, the method of quasi-induction (QI) has been validated as accessible to students and beneficial for their mastery of PMI (Cusi & Malara, 2008; Harel, 2002). However, a cognitive gap remains in transitioning between QI and formal PMI. Therefore, Cycle 1 centered around understanding the factors contributing to this gap and designing instructional materials that help students address it.

Preparing for the Experiment

Informed by prior research, we designed a set of scenarios about an unspecified proposition $P(n)$ (Table 1). For each scenario, students are tasked to decide whether the given information is sufficient to prove that $P(n)$ is true for all positive integers n. The tasks are independent of mathematical content. They are built from students' conceptualizations of logical implication, and they aim to bridge the gap between QI and PMI. We first validated these tasks in Arnold and Norton (2017) and Norton and Arnold (2017) via clinical interviews with students from an introductory proofs course, and we further refined them in a follow-up study (Norton & Arnold, 2019).

Table 1. Cycle 1 tasks.

Suppose $P(n)$ is a statement about a positive integer n, and we want to prove the claim that $P(n)$ is true for all positive integers n. For each scenario, decide whether the given information is enough to prove $P(n)$ for all positive integers n.	
A	$P(1)$ is true; for all integers $k \geq 1, P(k)$ is true.
B	$P(1)$ is true; there is an integer $k \geq 1$ such that $P(k)$ is true.
C	$P(1)$ is true; there is an integer $k \geq 1$ such that $P(k) \rightarrow P(k+1)$.
D	$P(1)$ is true; for all integers $k \geq 1$, $P(k) \rightarrow P(k+1)$.
E	For all integers $k \geq 1$, $P(k) \rightarrow P(k+1)$.
F	For all integers $k \geq 1$, $P(k)$ and $P(k+1)$ are true.
G	$P(1)$ is true; for all integers $k \geq 2$, $P(k) \rightarrow P(k+1)$.

Conducting the Experiment

We implemented research-based instruction in two sections of Introduction to Proofs at a large public university in the southeastern United States. The third author used the tasks within an informal pre-assessment. The fourth author used them to generate class discussion. Following these approaches, both instructors sought to promote the inductive implication as an invariant relationship between the inductive assumption and the inductive step (rather than treating the inductive assumption and step as separate components), and to explicitly address the issues of quantification. All classes associated with instruction on PMI were recorded. The third author had three class meetings, 50 minutes each. The fourth author held three meetings, 75 minutes each. Recordings captured the instructors' activity, the notes and PowerPoint slides displayed on the overhead projector, and students' interactions.

Retrospective Analysis

We performed two rounds of retrospective analysis of classroom data. During the first round, we coded the data using a set of codes informed by existing literature on students' struggles with PMI. They included base case (B), quantifiers (Q), inductive implication (I), domain knowledge (D), and conflation (C) between the inductive hypothesis, implication, or proposition. As new themes emerged for which we did not have an existing code, a new code was created. In this case, we reviewed and re-coded the prior data, using a constant comparative method (Glaser, 1965). This iterative process repeated until the codebook stabilized. The new codes included reducing the problem to computational setting (R_c), "why k, not n?" (K), effects of formal instruction (F), and the cognitive gap (G).[1]

We conducted the second round of analysis with an emphasis on the cognitive gap. Specifically, we reanalyzed the data to document how each code was related to the gap. As a byproduct of this analysis, two more codes emerged – intellectual need (N), and the use of language for communicating the intended logic (L).

The analysis revealed five themes describing the relationships between the cognitive gap and other challenges, depicted in Figure 2. Students must develop an intellectual need to generalize the logic involved in building the quasi-inductive chain of inferences (Theme 1). However, even when students have an intellectual need for formal PMI, issues with quantification arise (Theme 2). These challenges include careful quantification of the inductive implication and a shift in the language for quantifying the inductive assumption when proving the implication. As a consequence, students require new language to support the distinctions they make in quantifying the inductive assumption, the inductive implication, and the proposition they must prove (Theme 3). Finally, we found that a shift in notation from n to k when denoting the inductive implication seemed unnecessary to students. Students did not demonstrate the intellectual need for it in either class (Theme 5), nor for the use of proper language for quantifying n and k (Theme 4).

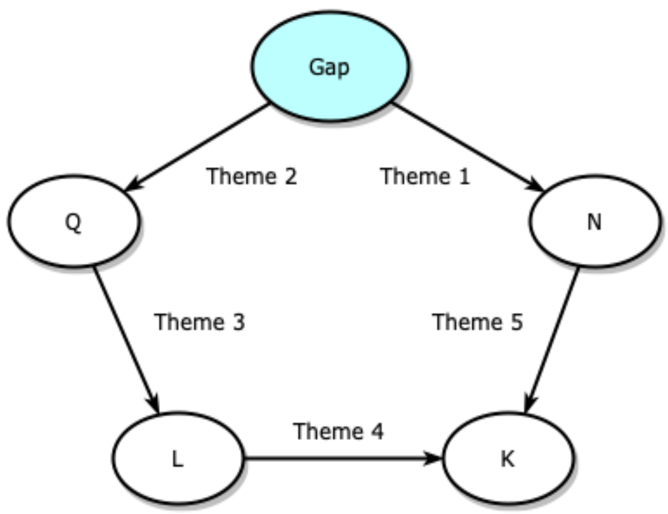

Figure 2. Five themes emerged from Cycle 1.

Cycle 2 (Summer 2022-Spring 2023)

Cycle 2 is ongoing. Its purpose is to broaden the scope of our study to the fundamental topics traditionally studied in introductory proofs courses, such as mathematical statements, logical implications and their transformations, quantifiers, and functions.

Preparing for the Experiment

Cycle 2 began with an advisory board meeting in June 2022. The advisory board members included three widely recognized experts in research on PMI, quantifiers, and logical reasoning. We relied on their expertise, on the previous research findings, and on the results from Cycle 1,

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[1] A detailed description of the codes can be found in Norton et al. (2022).

to 1) identify the EOs associated with teaching and learning proofs, and 2) design instructional tasks aimed at evoking, addressing, and assessing the discussed EOs.

In particular, as one of the outcomes of Cycle 1, we found that students' apparent treatment of inductive implication is closely linked with a language issue inherent in quantifying an implication. On the one hand, the implication $P(k) \to P(k+1)$ must be proved *for all* values of k. On the other hand, the proof of the implication requires one to assume $P(k)$ is true *for some fixed but arbitrary* value of k. Ignoring the former quantification leads to an incomplete argument. Ignoring the latter quantification leads to circular reasoning. Similar issues with language in quantifying mathematical objects have been documented in prior research (Dawkins & Roh, 2020; Lew & Mejía-Ramos, 2019).

To evoke and address challenges related to teaching and learning quantifiers, we designed six multiply quantified statements about a linear equation $mx + b = 0$ (see Table 2). The statements exhaust all combinations of ordering explicit quantifiers and their attached variables that make sense geometrically. Statement 2 is the only false statement. The variety of statements will allow students to compare and contrast the quantifiers in a different order, and, as a result, understand the role of quantifiers in mathematical statements.

Table 2. A sample Cycle 2 task.

Below are six statements about real numbers *m, x,* and *b.*
TASK 1: give a geometric interpretation to each of these statements. One of them is not true. Which one? Explain why this statement is false.
TASK 2: Match the true statements with the following proofs:

Statement	**Proof**
The exist real numbers *m,b,* and *x,* such that $mx + b = 0$.	Let $b = 0, m = 1$, and $x = 0$. Therefore, $mx + b = 1 \cdot 0 + 0$.
For all real numbers *m,b,* and *x,* $mx + b = 0$.	N/A
There exists a real number *m,* such that for all real numbers *b,* there exists a real number *x,* such that $mx + b = 0$.	Let $m = 1$. Then, for any real number *b,* take $x = -b$. Therefore, $mx + b = 1 \cdot (-b) + b = -b + b = 0$.
There exists a real number *m,* such that for all real numbers x, there exists a real number *b,* such that $mx + b = 0$.	Let $m = 1$. Then, for any real number *x,* take $b = -x$. Therefore, $mx + b = 1 \cdot (-x) + x = -x + x = 0$.
There exists a real number *b,* and there exists a real number *m,* such that for all real numbers *x,* $mx + b = 0$.	Fix an arbitrary *x.* Let $b = 0$. Observe that, when $m = 0$, $mx + b = 0 \cdot (-x) + 0 = 0 + 0 = 0$.
There exists a real number *b,* and there exists a real number *x,* such that for all real numbers *m,* $mx + b = 0$.	Let $b = 0$. Observe that, when $x = 0$, $mx + b = m \cdot 0 + 0 = 0 + 0 = 0$ for any real number *m.*

Conducting the Experiment

As in Cycle 1, the third and the fourth authors will implement research-based instruction in their respective Introduction to Proofs classes: one in Fall 2022 and one in Spring 2023. The classroom interactions pertaining to the aforementioned topics will be video and audio recorded.

Retrospective Analysis

We will qualitatively analyze the complex student-teacher and student-student interactions. The analysis will be reminiscent of the procedures we used in Cycle 1, involving the elements of constant comparative method (Glaser, 1965). To present the data in a comprehensive and feasible fashion, we will create a graph of connected codes, in which the nodes and edges will respectively represent the key codes that pertain to answering the research questions and the relationships between these codes, respectively.

Implications and Products

We will use the results of the first two cycles of our project to deepen our understanding of the EOs and disseminate research findings. Specifically, throughout the research-based instruction, we will collect data for a phenomenographic study to gain more nuanced insights into how students experience EOs during instructional interactions. In parallel with Cycle 2 data analysis, we will recruit instructors teaching Introduction to Proofs courses in Fall 2023 who are willing to participate in a week-long workshop preceding the fall semester and implement the proposed instruction. The workshop will be organized by the authors and will include selected videos of instructional interactions from Cycles 1 and 2. We will also conduct a phenomenographic study to see how the instructors experience the proposed EOs.

On the basis of the data collected in Cycles 1 and 2, we will compile an online handbook documenting the instructional interactions surrounding the EOs. The handbook will include our modified instructional tasks designed for evoking these interactions, as well as video clips of classroom interactions from the Introduction to Proofs classes taught by the third and fourth authors. The handbook is intended to increase instructors' pedagogical content knowledge on foundational topics of the course. In addition to serving as an instructional guide for bridging research and practice, the handbook will also provide instructional videos that can be shared directly with students.

To bolster the broader impacts of our project, we will create a "Train-the-Trainers" style professional development website. This website will transform and extend the training activities of our workshop (organized in the summer of 2023) into professional development activities that can be hosted and facilitated by any university wishing to implement the instructional tools resulting from this project. Each activity will be accompanied by a facilitator's guide for empowering and equipping members of other universities to conduct their own in-house training workshop for the instructors of their introductory proofs courses.

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